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Mathematics

Limits of zeroes of recursively defined polynomials

(linear homogeneous recursion/chromatic polynomial)

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ABSTRACT Let $\{P_n(z)\}$ be a sequence of polynomials satisfying a linear homogeneous recursion whose coefficients are polynomials in z. Necessary and sufficient conditions are found, subject to mild nondegeneracy conditions, that a number x be a limit of zeroes of $\{P_n\}$ in the sense that there is a sequence $\{z_n\}$ with $P_n(z_n) = 0$, $z_n \to x$. An application is given to a family of polynomials arising in a map-coloring problem.

1. Main result

Let $\{P_n(z)\}$ be a sequence of polynomials satisfying

$$P_{n+k}(z) = -\sum_{j=1}^{k} f_j(z)P_{n+k-j}(z). \qquad [1]$$

where the f_j are polynomials.

The complex number x is said to be a *limit of zeroes* of $\{P_n\}$ if there is a sequence $\{z_n\}$ such that $P_n(z_n) = 0$ and $z_n \to x$. Our main result is a necessary and sufficient condition, subject to two mild nondegeneracy conditions, that x be a limit of zeroes of $\{P_n\}$.

For fixed z, the solution of [1] given $P_0(z), \ldots, P_{k-1}(z)$ depends on the roots of the characteristic equation

$$\lambda^k + \sum_{j=1}^{k} f_j(z)\lambda^{k-j} = 0.$$

The roots, $\lambda_j(z)$, $j = 1, \ldots, k$, are algebraic functions of z, and for any z such that the $\lambda_j(z)$ are distinct,

$$P_n(z) = \sum_{j=1}^{k} \alpha_j(z)\lambda_j(z)^n. \qquad [2]$$

where the α_j are determined by solving the system of equations obtained by letting $n = 0, 1, \ldots, k-1$ in [2].

The nondegeneracy conditions needed for our result are

A. $\{P_n\}$ satisfies no recursion of order less than k.

B. There does not exist a constant ω with $|\omega| = 1$ and $\lambda_i(z) \equiv \omega\lambda_j(z)$ for some $i \neq j$.

THEOREM. *Suppose that $\{P_n\}$ satisfies [1], A and B. Then x is a limit of zeroes of $\{P_n\}$ if and only if the roots can be numbered so that one of the following holds:*

$$\text{(i) } |\lambda_1(x)| > |\lambda_j(x)|, 2 \leq j \leq k, \text{ and } \alpha_1(x) = 0$$

$$\text{(ii) } |\lambda_1(x)| = |\lambda_2(x)| = \ldots = |\lambda_l(x)| > |\lambda_j(x)|,$$
$$l + 1 \leq j \leq k, \text{ for some } l \geq 2.$$

2. Comments

A bare outline of the proof, details of which will appear elsewhere, is as follows. Suppose, typically, that x is such that the $\lambda_j(x)$ are distinct, so that the same is true in a neighborhood of x. Referring to [2], it is routine to show that if x is a limit of zeroes and (ii) fails to hold, then (i) must hold. The proof of sufficiency involves showing that if (i) or (ii) holds, then in any neighborhood of x, $Q_n(z) = P_n(z)/\lambda_1(z)$ and perforce $P_n(z)$ has a zero for all sufficiently large n. In each case, a winding number argument is used; if (i) holds, Rouche's theorem suffices, while if (ii) holds, a more complicated method is needed. If the $\lambda_j(x)$ are not distinct, complications arise which are essentially technical.

If the condition A does not hold, the *Theorem* can be applied to the unique lowest-order recursion satisfied by $\{P_n\}$. The situation is more interesting when B fails to hold. Variants of (i) and (ii) exist such that a limit of zeroes must satisfy one or the other. As for the question of the sufficiency of these variants, suffice it to say that if $\lambda_1(z) \equiv \omega\lambda_2(z)$, $\omega = e^{2\pi i\theta}$, and the other λ_j satisfy B, then the answer depends on whether θ is integral, rational but nonintegral, or irrational. Moreover, examples show that in the last case the answer seems to depend upon the degree of transcendence of θ.

3. An application

Given a map M, a coloring of M is an assignment of a color to each region of M such that contiguous regions are colored differently. For a positive integer m, let $Q(M,m)$ be the number of ways M can be colored with m or fewer colors, regarding as distinct even those colorings which can be obtained from each other by permutations of the colors. G. D. Birkhoff showed (ref. 1) that there is a polynomial P_M such that $Q(M,m) = P_M(m)$ for all positive integers m. A polynomial arising in this way is called a chromatic polynomial; the four-color conjecture amounts to the nonexistence of a chromatic polynomial with a zero at 4.

In a paper submitted to the *Journal of Combinatorial Theory,* the first two authors consider a sequence of maps $\{M_n\}$ consisting of an inner region and an outer region separated by n rings, each containing four regions. It is found that the corresponding sequence $\{P_n\}$ of chromatic polynomials satisfies a recursion of the form [1], with $k = 2$,

$$f_1(z) = -(z-3)(z^3 - 9z^2 + 33z - 48),$$
$$f_2(z) = 2(z-3)^3(z-2)(z^2 - 5z + 5).$$

When $z = 4$, the characteristic equation has 2 as a repeated root, implying in particular that (ii) holds so that 4 is a limit of zeroes of $\{P_n\}$. [Moreover, the only other real numbers which are limits of zeroes of $\{P_n\}$ are 0, 1, 2, 3, and $\frac{1}{2}(3 + \sqrt{5})$.]

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1. Birkhoff, G. D. (1912) "A determinant formula for the number of ways of coloring a map," *Ann. Math.* (2) 14, 42–46.